Situation: Zero Exponents

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Prompt

In an Algebra I class, a student questions the claim that *$a^0 = 1$ for all non-zero real number values of a.* The student asks, "How can that be possible? I know that a^0 is *a* times itself zero times, so a^0 must be zero."

Commentary

Mathematical Foci

Mathematical Focus 1

This scenario can be explained by using the multiplication and division properties of exponents. For example, the expression a^0 is equivalent to the expression $\frac{a^n}{a^n}$. Using the division property of exponents, $\frac{a^n}{a^n}$ is equivalent to a^{n-n} or a^0. We know that $\frac{a^n}{a^n} = 1$ because of the multiplicative identity field property. Therefore, because of the transitive property, a^0 must equal 1. We also know that 0^n=0. If a=0, then $\frac{a^n}{a^n} = \frac{0}{0}$, in an indeterminate form.

Using the multiplication property of exponents, a^0 or a^{n+-n} is equivalent to $a^n \cdot a^{-n}$. . If $a \neq 0$, then $a^n \neq 0$ and there is a number, a^{-n}, that is the multiplicative inverse of a^n. We know that $a^n \cdot a^{-n} = 1$because the product of multiplicative inverses is the multiplicative identity, 1. Therefore, because of the transitive property, a^0 must equal 1.

Mathematical Focus 2

It could be helpful to look at a pattern involving the recursive nature of exponential growth in order to explore this question. First consider a specific example using exponents with base 4.

$$4^{-3} = \frac{1}{4^3} = \frac{1}{64}$$

$$4^{-2} = \frac{1}{4^2} = \frac{1}{16}$$

$$4^{-1} = \frac{1}{4^1} = \frac{1}{4}$$

$$4^0 = ?$$

$$4^1 = 4 = 4$$

$$4^2 = 4 \cdot 4 = 16$$

$$4^3 = 4 \cdot 4 \cdot 4 = 64$$

As the exponent increases by 1, each successive term can be obtained by multiplying the preceding term by 4. That is, $4^{n+1} = 4 \cdot 4^n$. In order for this recursive pattern to hold for all integer values of *n*, 4^0 must be equal to 1.

This pattern can be generalized to all positive values of *a*. Consider the following table.

$$a^{-3} = \frac{1}{a^3}$$

$$a^{-2} = \frac{1}{a^2}$$

$$a^{-1} = \frac{1}{a^1}$$

$$a^0 = 1$$

$$a^1 = a$$

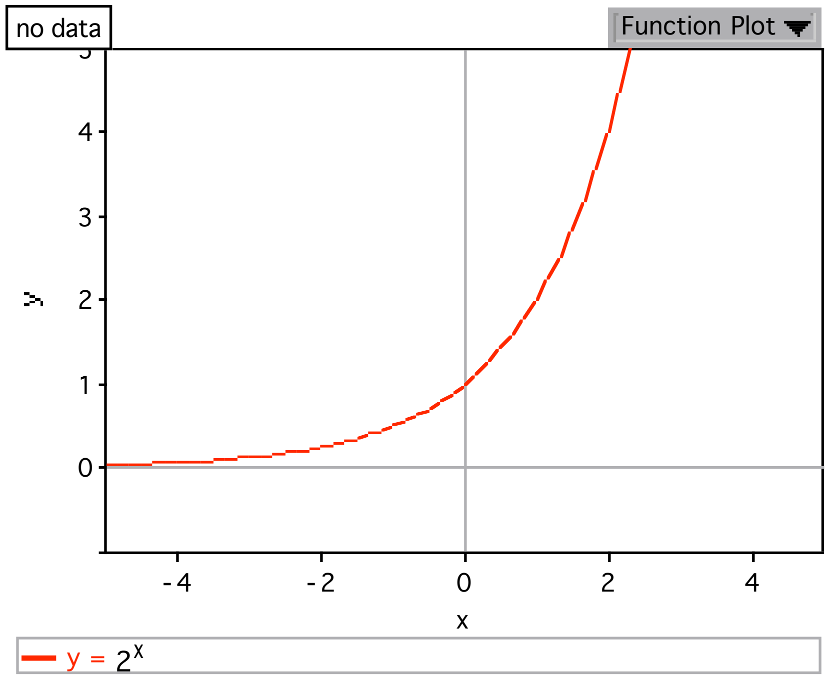
$$a^2 = a \cdot a$$

$$a^3 = a \cdot a \cdot a$$

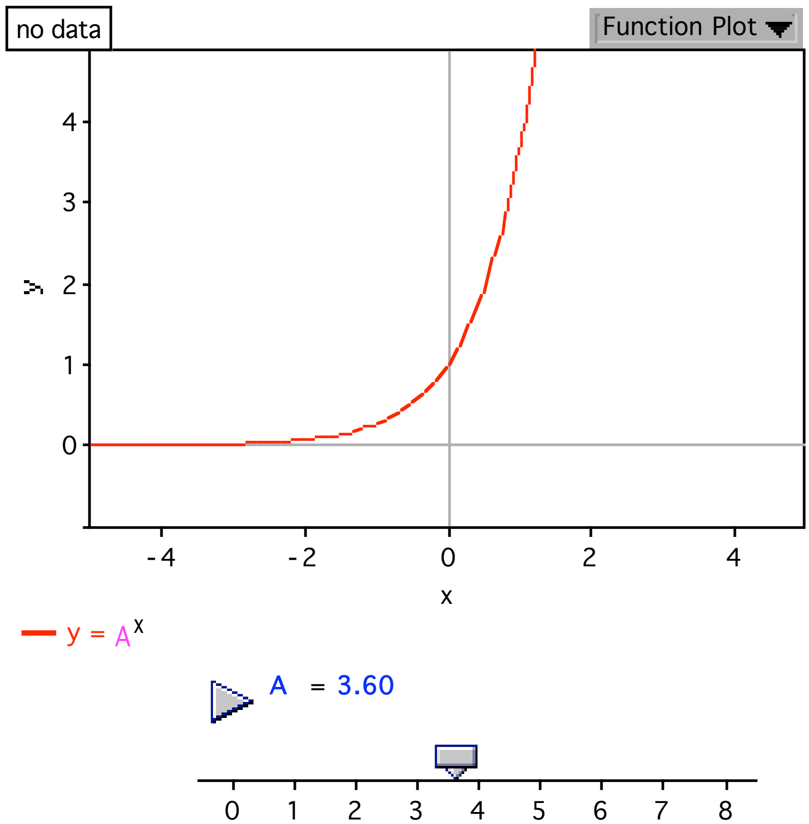
We can verify that the pattern holds by looking at properties of exponents. When *n* is a whole number greater than or equal to 1, then a^n is *a* used as a factor *n* times and a^{n+1} is *a* used as a factor *n+1* times, which is the same as *a* times the result of *a* used as a factor *n* times. So, for all positive values of *a,* $a^{n+1} = a \cdot a^n$. When n = 0, then $a^{0+1} = a \cdot a^0$. Since $a^{0+1} = a^1 = a$, it follows then that a^0 must be equal to 1.

Mathematical Focus 3

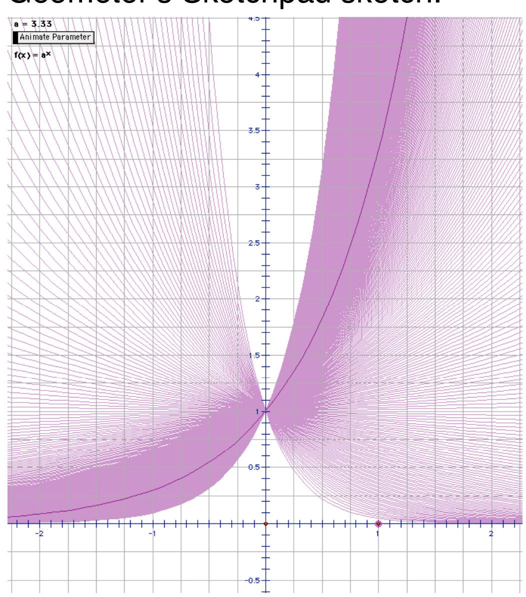
Another approach to explore this problem is through a graphical representation of the function $y = a^x$ for various real values of *a*. The following graph depicts $y = 2^x$. The value of 2^0 can be interpolated from the graph.



This behavior of $y = a^x$ at *x* = 0 can be explored graphically for several positive values of *a*. This can be investigated dynamically using the slider feature in Fathom. The graph below represents $y = a^x$ where the value of *a* is indicated by the slider. In this case *a* = 3.60. As the value on the slider is changed, the graph is updated automatically to reflect the change. The value $a^0 = 1$ can be interpolated from the graph for any positive value of *a*



. In fact, the point (a^n. 1) is the common point for graphs of functions given by $y = a^x$, a>0, as we can see when the graphs of $y = a^x$ for positive values of a are traced in the following Geometer's Sketchpad sketch.



Mathematical Focus 4

First consider the function $f(x) = a^x$ for a specific value of *a*, say *a = 2*. This function is continuous over all real x; therefore $f(0)$ will be defined. To determine $f(0)$, consider $\lim_{x \to 0}(2^x)$. To estimate $\lim_{x \to 0}(2^x)$ numerically, examine values of $f(x) = 2^x$ near $x = 0$.

x	2^x
-0.0004	0.99972278
-0.0003	0.999792077
-0.0002	0.99986138
-0.0001	0.999930688
0	1
0.0001	1.000069317
0.0002	1.000138639
0.0003	1.000207966
0.0004	1.000277297

As the values of x approach zero, the values of $f(x) = 2^x$ approach 1; therefore, it appears that $\lim_{x \to 0}(2^x) = 1$. This procedure can be expanded to all positive values for *a*.

Using the definition of limit, you can also find the value *a*^{0}. Since the function $f(x) = a^x$ is continuous over all numbers, we know that the value of the function $f(x) = a^x$ at x = 0 is equal to $\lim_{x \to 0}(a^x)$.

To prove that $\lim_{x \to 0}(a^x) = 1$, show that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|a^x - 1| < \varepsilon$ when $0 < |x - 0| < \delta$. For all x in the interval (-1,1) we know that $|a^x - 1| < 1$. If $|a^x - 1| < 1$, it follows that $x < 1$. Therefore $\delta < 1$ and $\lim_{x \to 0}(a^x) = 1$.

Mathematical Focus 5

Consider the case when *a* is the imaginary unit, then exponents can be expressed as rotations around the unit circle rather than as repeated multiplication. Begin with the imaginary unit raised to an integer power. Using the definition of i, $i = \sqrt{-1}$, then the following hold:

$$i^{-4} = 1$$
$$i^{-3} = i$$
$$i^{-2} = -1$$
$$i^{-1} = -i$$
$$i^{0} = ?$$
$$i^{1} = i$$
$$i^{2} = -1$$
$$i^{3} = -i$$
$$i^{4} = 1$$
$$i^{5} = i$$

The powers of the imaginary unit rotate around the unit circle on the complex plane. Therefore, since $i^1 = i$ and $i^4 = 1$, it follows that $i^0 = 1$.

Mathematical Focus 6

Consider DeMoivre's Theorem, $e^{i\theta} = \cos\theta + i\sin\theta$. Let $\theta = 0$, then $e^{i\cdot 0} = \cos 0 + i\sin 0$ and $e^0 = 1$. A geometric result of DeMoivre's theorem is that the values of $e^{i\theta}$ will designate locations on the unit circle mapped in the complex plane. DeMoivre's theorem can be extended to $a^{i\theta}$, where $a^{i\theta} = \left(e^{(\ln a)}\right)^{i\theta}$, therefore when *a* is raised to the bi power, where b is a real constant and i is the imaginary unit, it will designate a location on the unit circle in the complex plane. Therefore, it would follow that a^{0i} should also designate a location on the unit circle in the complex plane. Since a^{0i} only contains a real part, it must be located on the x-axis. Moreover, since *a* raised to a real power must be positive, a^{0i} must equal 1.

References

Lakoff, George; Nunez, Rafael; Where Mathematics Comes From: How the Embodied Mind Brings Math into Being; 2001

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